Transverse Electrical (TE) Field Modes in an Asymmetric Slab Waveguide

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(Student Handout)
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1. Geometry And Notation

We consider a **slab waveguide** that is uniform in y and z and varies only in the transverse coordinate x:

$$n(x) = \begin{cases} n_{\text{bot}}, & x < -d/2, \\ n_{\text{core}}, & |x| \le d/2, \\ n_{\text{top}}, & x > d/2, \end{cases} \qquad n_{\text{core}} > \max(n_{\text{top}}, n_{\text{bot}}).$$

The optical field propagates along z with propagation constant β. The vacuum wavenumber is

$$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

2. From Maxwell To The Scalar Wave Equation

In a source-free, non-magnetic dielectric ($\mu = \mu_0$) with time-harmonic fields $\propto e^{-i\omega t}$, the electric field satisfies the vector wave equation

$$\nabla^2 \mathbf{E} + k_0^2 n(x)^2 \, \mathbf{E} = 0.$$

For a slab, we choose **TE polarization** (electric field along y):

$$\mathbf{E}(x, z, t) = \hat{\mathbf{y}} \, E_y(x) \, e^{i\beta z} \, e^{-i\omega t}.$$

There is no y dependence, so $\partial/\partial y = 0$.

3. Separation Of Variables $\Rightarrow$ A 1D (Transverse) ODE

Insert $E_y(x)e^{i\beta z}$ into the scalar wave equation for the y component:

$$\frac{\partial^2}{\partial x^2}\left(E_y e^{i\beta z}\right) + \frac{\partial^2}{\partial z^2}\left(E_y e^{i\beta z}\right) + k_0^2 n(x)^2 \left(E_y e^{i\beta z}\right) = 0.$$

Compute derivatives:

$$\frac{\partial^2}{\partial x^2}\left(E_y e^{i\beta z}\right) = E_y''(x) e^{i\beta z}, \qquad \frac{\partial^2}{\partial z^2}\left(E_y e^{i\beta z}\right) = -\beta^2 E_y(x) e^{i\beta z}.$$

Cancel the common factor $e^{i\beta z}$:

$$\boxed{E_y''(x) + \Big(k_0^2 n(x)^2 - \beta^2\Big) E_y(x) = 0.}$$

This is the **transverse mode equation**. It is already an "eigenvalue-style" equation: the allowed values of β are determined by boundary conditions.

4. From ODE To Matrix Eigenvalue Problem

Rearrange the transverse equation:

$$\boxed{\Big(\frac{d^2}{dx^2} + k_0^2 n(x)^2\Big) E_y(x) = \beta^2\, E_y(x).}$$

Interpretation:

- $E_y(x)$ is the **eigenfunction** (mode profile).
- β^2 is the **eigenvalue** (square of propagation constant).

Numerical Discretization (Finite Differences, FD)

On a grid x_i, approximate d^2/dx^2 by a finite-difference matrix D_{xx}. Then

$$\boxed{\Big(D_{xx} + \mathrm{diag}(k_0^2 n(x_i)^2)\Big)\mathbf{E} = \beta^2\,\mathbf{E}.}$$

Solving this matrix eigenproblem yields discrete pairs (β_m, E_m).

5. The Guide in Waveguide

Define the core wavenumber magnitude $k_{\text{core}} = k_0 n_{\text{core}}$. With $k_z = \beta$,

$$k_x = \sqrt{k_{\text{core}}^2 - \beta^2}.$$

In each *uniform* region (constant n), the mode equation becomes:

$$E'' + (k_0^2 n^2 - \beta^2)E = 0.$$

Core ($n = n_{\text{core}}$): if $\beta < k_0 n_{\text{core}}$ then k_x is real and the field is oscillatory.

Cladding j ($n = n_j$): define

$$\alpha_j = \sqrt{\beta^2 - (k_0 n_j)^2}.$$

If $\beta > k_0 n_j$, then α_j is real and the field decays as $E \sim e^{-\alpha_j |x|}$ (evanescent tail).

6. Cutoff In An (Asymmetric) Slab

For a mode to be guided, it must be evanescent in *both* claddings:

$$\beta > k_0 n_{\text{top}} \quad \text{and} \quad \beta > k_0 n_{\text{bot}}.$$

Therefore the guided-mode interval is

$$\boxed{k_0\, n_{\text{clad,max}} < \beta < k_0\, n_{\text{core}}, \qquad n_{\text{clad,max}} = \max(n_{\text{top}}, n_{\text{bot}}).}$$

Cutoff occurs when the evanescent decay on the *higher-index* cladding side vanishes:

$$\boxed{\beta_{\text{cutoff}} = k_0\, n_{\text{clad,max}}.}$$

Equivalently, in terms of **effective index**

$$n_{\text{eff}} = \frac{\beta}{k_0}, \qquad \boxed{n_{\text{eff,cutoff}} = n_{\text{clad,max}}.}$$

7. Eigenvalue Spectrum And Cutoff

Suppose your numerical solver returns eigenvalues $\{\beta_m\}$ (sorted from largest to smallest). Then:

1. Compute $n_{\text{clad,max}} = \max(n_{\text{top}}, n_{\text{bot}})$ and $\beta_{\text{cutoff}} = k_0 n_{\text{clad,max}}$.
2. Filter guided modes: keep only β_m such that

$$\beta_{\text{cutoff}} < \beta_m < k_0 n_{\text{core}}.$$

3. For each guided mode compute distance to cutoff:

$$\Delta\beta_m = \beta_m - \beta_{\text{cutoff}} > 0.$$

4. The **last guided mode (nearest cutoff)** is the one with the *smallest* positive $\Delta\beta_m$.

As you reduce the core thickness d (or reduce index contrast), this highest-order guided mode is the first to disappear.

8. Connection To "Root-Finding"

In a piecewise-constant slab, one can enforce continuity of E_y and dE_y/dx at $x = \pm d/2$, leading to a transcendental equation for β.

A numerically stable TE form is the **phase condition**:

$$\boxed{k_x d - \arctan\!\left(\frac{\alpha_{\text{top}}}{k_x}\right) - \arctan\!\left(\frac{\alpha_{\text{bot}}}{k_x}\right) = m\pi, \qquad m = 0, 1, 2, \ldots}$$

where $k_x = \sqrt{(k_0 n_{\text{core}})^2 - \beta^2}$ and $\alpha_j = \sqrt{\beta^2 - (k_0 n_j)^2}$. This must be solved by **root finding** in β for each integer m.

9. Cutoff Angle

Inside the core, the wavevector has magnitude $|\mathbf{k}| = k_0 n_{\text{core}}$ and components (k_x, k_z) with $k_z = \beta$.

Angle from the interface normal (x-axis) (TIR convention):

$$\boxed{\theta_{\text{normal}} = \arcsin\left(\frac{k_z}{|\mathbf{k}|}\right) = \arcsin\left(\frac{\beta}{k_0 n_{\text{core}}}\right) = \arcsin\left(\frac{n_{\text{eff}}}{n_{\text{core}}}\right).}$$

This is the angle to compare against critical angles for total internal reflection.

Angle from the propagation axis (z-axis):

$$\boxed{\theta_{\text{axis}} = \arccos\left(\frac{\beta}{k_0 n_{\text{core}}}\right) = 90^\circ - \theta_{\text{normal}}.}$$

Critical angles (from the normal):

$$\theta_{c,\text{top}} = \arcsin\left(\frac{n_{\text{top}}}{n_{\text{core}}}\right), \qquad \theta_{c,\text{bot}} = \arcsin\left(\frac{n_{\text{bot}}}{n_{\text{core}}}\right).$$

Guiding requires $\theta_{\text{normal}} > \max(\theta_{c,\text{top}}, \theta_{c,\text{bot}})$, which is equivalent to $\beta > k_0 n_{\text{clad,max}}$.

10. Summary

- Start from $\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} = 0$ and use $E_y(x)e^{i\beta z}$ to get
$$E_y'' + \left(k_0^2 n(x)^2 - \beta^2\right) E_y = 0.$$
- Rewrite as an eigenproblem:
$$\left(\frac{d^2}{dx^2} + k_0^2 n(x)^2\right) E_y = \beta^2 E_y.$$
- Guided modes satisfy $k_0 n_{\text{clad,max}} < \beta < k_0 n_{\text{core}}$.
- Cutoff occurs at $\beta_{\text{cutoff}} = k_0 n_{\text{clad,max}}$ (equivalently $n_{\text{eff}} = n_{\text{clad,max}}$).
- The last guided mode is the guided mode with smallest $\beta > \beta_{\text{cutoff}}$.