Trigonometry

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1 Motivation

In physics trigonometry has many applications. For example, trigonometry is critical for the understanding oscillatory and wave phenomena, where periodic functions (sine and cosine waves) describe oscillations and waves, such as those in sound, light, and electrical circuits. They help model how these systems evolve over time. Another example is rotational motion, here trigonometry enables the analysis of circular and rotational motion by relating angles to arc lengths and positions along a circular path.

2 Definition sin, cos, tan

sin, *cos*, and *tan* are defined through the relationship between angle and the sides of a right-angle triangle. The quantities needed to define the three trig functions are shown in the figure

$$cos\left(\alpha\right) = \frac{a}{c}$$
$$sin\left(\alpha\right) = \frac{b}{c}$$
$$tan\left(\alpha\right) = \frac{b}{a} = \frac{sin\left(\alpha\right)}{cos\left(\alpha\right)}$$

how the angles are defined depends on the setting on your calculator, it could be either in degrees, with 360^o being a full circle, or it could be in radians, with $2 \cdot \pi$ being a full circle. And the relationship between the two angle measures is given by

$$\alpha\left(degree\right) = \frac{180}{\pi}\alpha\left(radians\right)$$

3 Connection to Complex Numbers

As outlined in the primer on complex numbers, *sin* and *cos* are connected to a complex complex numbers through Euler's formula

$$
\begin{aligned}
e^{i\alpha} &= cos\left(\alpha\right) + i \cdot sin\left(\alpha\right) \\
cos\left(\alpha\right) &= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \\
sin\left(\alpha\right) &= \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)
\end{aligned}
$$

where $i = \sqrt{-1}$ is the imaginary unit.

4 Adding Angles and Other Identities

The value of Euler's formula becomes apparent when deriving trigonometric identities, for example for adding angles

$$cos\left(\alpha + \beta\right) = ?$$

we proceed as follows

$$
\begin{aligned}
cos\left(\alpha + \beta\right) + i \cdot sin\left(\alpha + \beta\right) \\
= e^{i(\alpha+\beta)} \\
= e^{i\alpha} \cdot e^{i\beta} \\
= \left(cos\left(\alpha\right) + i \cdot sin\left(\alpha\right)\right) \cdot \left(cos\left(\beta\right) + i \cdot sin\left(\beta\right)\right) \\
= cos\left(\alpha\right) cos\left(\beta\right) - sin\left(\alpha\right) sin\left(\beta\right) \\
+ i\left(cos\left(\alpha\right) sin\left(\beta\right) + sin\left(\alpha\right) sin\left(\beta\right)\right)
\end{aligned}
$$

and now we compare real and imaginary parts in the first and last formula to obtain

$$
\begin{aligned}
cos\left(\alpha + \beta\right) &= cos\left(\alpha\right) cos\left(\beta\right) - sin\left(\alpha\right) sin\left(\beta\right) \\
sin\left(\alpha + \beta\right) &= cos\left(\alpha\right) sin\left(\beta\right) + sin\left(\alpha\right) sin\left(\beta\right)
\end{aligned}
$$

here is another handy identity

$$
\begin{aligned}
sin^2\left(\alpha\right) + cos^2\left(\alpha\right) \\
= e^{i\alpha} \cdot e^{-i\alpha} \\
1
\end{aligned}
$$

an alternative to start with the Pythagorean theorem for a right angle triangle

$$
\begin{aligned}
c^2 &= a^2 + b^2 \\
1 &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\
1 &= cos^2\left(\alpha\right) + sin^2\left(\alpha\right)
\end{aligned}
$$

and similarly you can derive other trig identities. Here is a last example

$$
\begin{aligned}
& tan\left(\alpha/2\right) \\
&= \frac{sin\left(\alpha/2\right)}{cos\left(\alpha/2\right)} \\
&= \frac{\frac{1}{2i}\left(e^{i\alpha/2} - e^{-i\alpha/2}\right)}{\frac{1}{2}\left(e^{i\alpha/2} + e^{-i\alpha/2}\right)} \\
&= \frac{1}{i}\frac{\left(e^{i\alpha} - 1\right)}{\left(e^{i\alpha} + 1\right)} \\
&= \frac{1}{i}\frac{\left(e^{i\alpha} - 1\right)}{\left(e^{i\alpha} + 1\right)}\frac{\left(e^{-i\alpha} + 1\right)}{\left(e^{-i\alpha} + 1\right)} \\
&= \frac{1}{i}\frac{2isin\left(\alpha\right)}{\left(2 + 2cos\left(\alpha\right)\right)} \\
&= \frac{sin\left(\alpha\right)}{\left(1 + cos\left(\alpha\right)\right)}
\end{aligned}
$$

and we see that last line is real, in agreement with line 1. In conclusion we found repeatedly a close relationship between trigonometric functions and complex numbers, that provides a convenient conceptual framework to derive trigonometric identities.

5 Exercises

1. Compute $tan\left(45^o\right)$.
2. For $\alpha = 30^o$ and $\beta = 20^o$ show that the two sides of the addition theorem for cos are the same.
3. Proof $cos\left(45^o\right) = 1/\sqrt{2}$.
4. Proof the following identity: $cos\left(\alpha/2\right) = \pm\sqrt{\frac{1+cos(\alpha)}{2}}$.
5. Proof the following identity: $cos\left(3\alpha\right) = 4 \cdot cos^3\left(\alpha\right) - 3 \cdot cos\left(\alpha\right)$.