Taylor Expansion – Function Approximation

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1 Motivation

When you're faced with a complicated function in physics—say a potential energy curve, a complicated dispersion relation, or a trigonometric function in an equation of motion—it often helps to approximate it by a simple polynomial. The tool that lets us do this systematically is the Taylor expansion.

2 Taylor Expansion

The idea of a Taylor expansion is to provide a rational approach to approximate a complicated function with a polynomial.

$$f\left(x\right)=\sum_{n=0}^{\infty}a_{n}\left(x-x_{0}\right)^{n}$$

where we are interested in describing the function in a neighborhood of the reference point x_0, and we need to determine the coefficients a_i such that our polynomial approximates the function as well as possible. In the following we assume that the function $f(x)$ can be differentiated as many times as needed. To see we can obtain the coefficients lets see what happens if we set $x = x_0$

$$\begin{aligned} f\left(x_{0}\right) &= \sum_{n=0}^{\infty}a_{n}\left(x_{0}-x_{0}\right)^{n} \\ &= a_{0} \end{aligned}$$

and we have determined the zero order coefficient

$$a_{0}=f\left(x_{0}\right)$$

now we differentiate both sides (once) and obtain

$$f'\left(x\right)=\sum_{i=1}^{\infty}n\times a_{n}\times\left(x-x_{0}\right)^{n-1}$$

where the lower index limit was adjusted to reflect that the derivative of a function is zero. Now we evaluate this expression for $x = x_0$

$$f'(x_0) = \sum_{n=1}^{\infty} n \times a_i \times (x_0 - x_0)^{n-1}$$
$$1 \times a_1$$

and the linear coefficient can be obtained from the derivative

$$a_1 = f'(x_0)$$

which suggests (correctly) that all coefficients can be obtained from the derivatives of the function at the reference point of interest. And you can show that the general formula is

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and the Taylor expansion itself is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

if we truncated the Taylor expansion at some power, we obtain a Taylor polynomial. Since it is a finite approximation to a function it will in general not be a perfect match. We indicate this truncation error as follows

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \mathcal{O}\left(x^{N+1}\right)$$

where the truncation error,$\mathcal{O}\left(x^{N+1}\right)$, corresponds to the lowest non-vanishing order that we neglected in our truncated Taylor polynomial. For example

$$f(x) = \sum_{n=0}^{2} a_n (x - x_0)^n + \mathcal{O}\left(x^3\right)$$

if you can show that the x^3 term is absent in the Taylor polynomial than you need to adjust the power of the error to reflect this knowledge.

3 Taylor Expansion Examples

3.1 exp(x), about x_0=0

we follow the machinery detailed in the previous section to compute the Taylor expansion of

$$f(x) = exp(x)$$

compute the needed derivatives and find $(0! = 1)$

$$a_n = \frac{1}{n!}$$
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

as a sanity check we notice that $exp(0) = f(0) = 1$, as it must be. We also note that a_1 is positive, suggesting a positive slope at our reference point, consistent with the fact that $exp(x)$ has a positive slope everywhere.

3.2 sin(x) , about $x_0 = 0$

and here is the Taylor expansion for $sin(x)$ with a reference point $x_0 = 0$

$$f(x) = sin(x)$$

we find that all even powered derivatives are zero at $x_0 = 0$. Thus, only odd powers contribute

$$a_{2n} = 0$$
$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

we note that the $n = 0$ term corresponds to the linear term in the polynomial, and $sin(x)$ behaves about $x = 0$ approximately as a straight line with slope 1. The next term has a negative coefficient, consistent with the observation that $sin(x)$ turns over as the argument increases.

3.3 cos(x), about $x_0 = 0$

similarly, we find for $cos(x)$ with a reference point $x_0 = 0$

$$f(x) = cos(x)$$

and this time we find that all odd powered derivative are zero, leaving only odd powers in the Taylor expansion

$$a_{2n} = \frac{(-1)^n}{(2n)!}$$
$$a_{2n-1} = 0$$
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

3.4 Taylor Expansion and Symmetry

As we just saw, for $sin(x)$, an odd function ($f(-x) = -f(x)$), only odd powers contribute to the Taylor expansion, and for $cos(x)$, and even function ($f(-x) - f(x)$), only even powers appear. The observation holds for all odd functions about the reference point. First adjust the argument to account for arbitrary reference points, x_0, and define the deviation from the reference point:

$$\varepsilon = x - x_0$$

$$f(-\varepsilon) = -f(\varepsilon)$$
$$f(-\varepsilon) = \sum_{n=0}^{\infty} a_n (-\varepsilon)^n = -\sum_{n=0}^{\infty} a_n \varepsilon^n = -f(\varepsilon)$$
$$\sum_{n=0}^{\infty} a_n (-1)^n \varepsilon^n = -\sum_{n=0}^{\infty} a_n \varepsilon^n$$

and comparing powers one by one, we find that even powers on the left and right side have opposite signs, which can only be satisfied if the coefficient is zero. Thus, the only terms left will be odd powers, consistent with the explicit calculation we did before. You can use the same strategy to show that for even functions about the reference point only even monomial powers remain. Finally, in the Taylor expansion for $exp(x)$, a function of mixed odd/even character, all powers contribute to the Taylor polynomial.

4 Taylor Expansions in Physics

4.1 Pendulum

One of the most famous examples is the small angle approximation used to describe small amplitude oscillations of a pendulum. You can show that the potential energy of such a pendulum can be written as

$$\begin{aligned} V(\theta) &= mgL\,(1 - cos(\theta)) \\ &\sim \frac{1}{2} mgL\theta^2 \end{aligned}$$

in words, for small oscillations, the pendulum performs harmonic oscillations about its equilibrium point ($\theta = 0$). And for the force we obtain

$$\begin{aligned} F(\theta) &= -\frac{dV}{d\theta} \\ &= -mgLsin(\theta) \\ &= -mgL\theta \end{aligned}$$

a linear restoring force. I would like to emphasize that this is only true for small enough oscillations. We also note, that while it may require in numerical techniques for most potentials such as, $V(\theta)$, if we can justify the use of a Taylor expansion, we may be able to determine the dynamical system behavior in this regime analytically, as in the case of our pendulum example that reduces to the analytically solvable problem of a linear spring.

4.2 Potential Energy Wells

Lets consider a general potential well and its Taylor expansion

$$V(x) = V_0 + \frac{dV}{dx}(x - x_0) + \frac{1}{2}\frac{d^2V}{dx^2}(x - x_0)^2 + \cdots$$

If we are interested in the dynamical behavior about the equilibrium point, x_0, we notice that the force (by definition) at the equilibrium must disappear

$$0 = F(x) = -\frac{dV}{dx}$$

and the expression simplifies to

$$V(x) = V_0 + \frac{1}{2}\frac{d^2V}{dx^2}(x - x_0)^2 + \mathcal{O}\left((x - x_0)^3\right)$$

and neglecting higher order terms this is again nothing else than harmonic motion about equilibrium. And vice versa, if we can measure the frequency of oscillations, we can learn something about the potential energy landscape of the system. To be more specific we define $\varepsilon = x - x_0$ as the deviation from equilibrium, and we obtain for the force

$$F(\varepsilon) = -\frac{d^2V}{dx^2}\varepsilon$$

and from Newton's 2nd law we obtain

$$\begin{aligned}\frac{d^2\varepsilon}{dt^2} &= -\frac{1}{m}\frac{d^2V}{dx^2}\varepsilon \\ &= -\omega^2\varepsilon\end{aligned}$$

and we obtain the desired relationship between frequency of oscillation and potential energy variation

$$\omega = \sqrt{\frac{1}{m}\frac{d^2V}{dx^2}}$$

and as a sanity check you can confirm by dimensional analysis that the units are consistent.

5 Conclusion

- Taylor expansions let you replace complicated functions with simple polynomials valid near a point.
- Truncating after a few terms can give you excellent approximations for small deviations.
- In physics and engineering, Taylor expansion are widely used, for example in the harmonic approximation around equilibrium, and series expansions to approximate dynamical behavior of complicated interacting systems.
- In mathematics Taylor expansion are for example used to approximate solutions to differential equations.

6 Exercises

1. Consider the Taylor expansion of $sin(x)$ about $x_0 = 0$. How far from the reference point can you move such that the lowest order (linear) term describes sin(x) to within 1 and 0.1? Now repeat the exercise but retain the two lowest terms in the Taylor expansion (linear and cubic). Compare and discuss the value ranges you find.
2. Assume that the function $f(x)$ is a polynomial, what is its Taylor polynomial?
3. Proof that the Taylor expansion for an even function contains only even power monomials.
4. Compute the Taylor expansion of $tan(x)$. Do all powers occur in the final expression? Discuss.