Complex Numbers

March 24, 2025

1 Motivation

Consider the following quadratic equation

$$x^2 = 1$$

with the two solutions

$$\begin{aligned} x_\pm &= \pm\sqrt{1} \\ &= \pm 1 \end{aligned}$$

in this equation we have used the label $\pm$ to distinguish the two solutions. Now let's make a small change to this quadratic equation

$$x^2 = -1$$

and we proceed as before to obtain the two solutions

$$x_\pm = \pm\sqrt{-1}$$

but $\sqrt{-1}$ is not a real number. Thus, we need to extend the real number system such that we can capture this new type of solutions. This extension leads to the definition of complex numbers.

2 Algebraic Definition of Complex Numbers

Complex numbers extend the real numbers by defining the complex unit, i, as

$$i^2 = -1$$

with this definition we can write the solution of

$$\begin{aligned} x^2 &= -1 \\ x_\pm &= \pm\sqrt{-1} \\ &= \pm i \end{aligned}$$

a more complete algebraic definition of complex numbers is

$$z = a + bi$$

where z symbolizes the complex number, a and b are real numbers and i is the complex unit we just defined. Another but equivalent way to connect complex numbers and real numbers is to define real and imaginary parts of a complex number.

$$\begin{aligned} a &= Re\left(z\right) \\ b &= Im\left(z\right) \\ z &= Re\left(z\right) + Im\left(z\right) i \end{aligned}$$

Complex numbers can be added, subtracted, multiplied and divided just like real numbers, However, we have to pay close attention to the fact that $i^2 = -1$, which connects imaginary and real part. Let's look at an example:

$$\begin{aligned} z_1 &= 3 + 4i \\ z_2 &= 3 - 2i \end{aligned}$$

and we add two complex numbers by adding their real and imaginary parts separately

$$\begin{aligned} & z_1 + z_2 \\ &= (3 + 3) + (4 - 2)\, i \\ &= 6 + 2i \end{aligned}$$

and we subtract two complex numbers by subtracting their real and imaginary parts separately

$$\begin{aligned} & z_1 - z_2 \\ &= (3 - 3) + (4 + 2)\, i \\ &= 0 + 6i \end{aligned}$$

now let's multiply the two complex numbers

$$\begin{aligned} & z_1 \cdot z_2 \\ &= (3 + 4i) \cdot (3 - 2i) \\ &= 3 \cdot 3 + 3 \cdot (-2i) + (4i) \cdot 3 + (4i) \cdot (-2i) \\ &= 9 - 6i + 12i + 8 \\ &= 17 + 6i \end{aligned}$$

in preparation for dividing two complex numbers, let's introduce the complex conjugate to a complex number

$$\begin{aligned} z &= a + bi \\ \bar{z} &= a - bi \end{aligned}$$

where the over-bar signifies the complex conjugate, and it means to leave the real part unchanged and to change the sign of complex part of the complex number, for example for the number, z_2, define earlier

$$\begin{aligned} z_2 &= 3 - 2i \\ \bar{z}_2 &= 3 + 2i \end{aligned}$$

something peculiar happens when we multiply a complex number with its complex conjugate

$$\begin{aligned} &z_2 \cdot \bar{z}_2 \\ &= (3 - 2i) \cdot (3 + 2i) \\ &= 9 + 6i - 6i + 4 \\ &= 13 \end{aligned}$$

in this case the product is a real number. This turns out to be a general result

$$z \cdot \bar{z} = a^2 + b^2$$

a relationship that comes in handy to divide two complex numbers

$$\begin{aligned} &z_1/z_2 \\ &= \frac{3+4i}{3-2i} \\ &= \frac{3+4i}{3-2i} \cdot \frac{3+2i}{3+2i} \\ &= \frac{1}{13}(3+4i) \cdot (3+2i) \\ &= \frac{1}{13}(9 + 6i + 12i - 8) \\ &= \frac{1}{13}(1 + 18i) \end{aligned}$$

a few comments. On the third line we multiplied with "1" which resolved as the ratio of the complex conjugate of the complex number in the denominator

$$1 = \frac{3+2i}{3+2i}$$

the result of this choice is that the product in the denominator is of the form $z \cdot \bar{z}$ which, as we showed above, is a real number. Finally for the product in the product in the nominator we follow the rules for products of complex numbers, introduced above.

3 Polar Representation of Complex Numbers

Earlier we observed

$$z \cdot \bar{z} = a^2 + b^2$$

This expression reminds us of the Pythagorean theorem, that connects the three sides of right angle triangle. Following this insight, we realize that we can reproduce this relationship if we visualize complex numbers in a 2-dimensional coordinate system, where the x-axis corresponds to the real part and the y-axis corresponds to the imaginary part of a complex number. We enter a complex number in this diagram by moving a units to the right turning 90^o and move b units up. An alternative to characterize the same triangle is to use the angle θ at the origin to write

$$\begin{aligned} z &= a + bi \\ &= r \cdot cos\left(\theta\right) + r \cdot sin\left(\theta\right) i \\ r &= a^2 + b^2 \end{aligned}$$

where in the last line we have defined r as the length of the hypotenuse of the triangle. It turns out that the second formula can be written in a short form, using Euler's identity

$$e^{i\theta} = cos\left(\theta\right) + sin\left(\theta\right) i$$

and the polar form of the complex number is

$$z = r \cdot e^{i\theta}$$

4 Applications

Complex numbers find a wide range of applications in Science and Engineering.

- Mechanics: the time dependence of harmonic motion can be written in compact form.
- Electricity and Magnetism: magnetic dipoles and field lines.
- Electricity and Magnetism: inductors in electric circuits.
- Quantum Mechanics: amplitude of state vectors.

A final note, using complex numbers to solve problems in mechanics and electricity and magnetism is a matter of convenience and you might equally well solve the problems using real valued calculus. In quantum mechanics this is no longer a matter of convenience, complex numbers become part of the fabric of nature, suggesting that quantum mechanics is fundamentally different from classical physics, with phenomena that have no classical counterpart. However, at the same time, classical physics must appear as some limit of quantum mechanics.

This brings us to the end of the primer on complex numbers.

5 Exercises

1. For the two complex numbers, $z_1 = 1 + i$ and $z_2 = 1 - 3i$, compute $z_1 \cdot z_2$.
2. For the two complex numbers, $z_1 = 1 + i$ and $z_2 = 1 - 3i$, compute z_1/z_2.
3. For the complex number, $z_1 = 1 + i$, compute $z_1 \cdot \bar{z}_1$.
4. Determine the component representation of $z_1 = 3 \cdot e^{i3\pi/2}$.
5. Determine the polar representation of $z_1 = 1 + i$.
6. Here is a challenge problem: Solve $1^x = 2$ for x.