From Generators to Fock-Space Superpositions: Coherent and Squeezed Light
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1 Introduction

Two of the most important single-mode optical states are (i) coherent states, which model ideal laser light as a displaced vacuum with a well-defined complex field amplitude, and (ii) squeezed vacuum states, which redistribute quantum fluctuations between conjugate quadratures while remaining minimum-uncertainty states with applications for example in interferometric noise reduction [1]. Both optical states are produced by simple exponentiated *generators*. This short primer shows, using only commutators and Taylor expansions, how the generator forms of the displacement and squeezing operators lead to explicit superpositions in the Fock (photon-number) basis [2].

2 Single-Mode Algebra

We consider one bosonic mode with annihilation (a) and creation ($a^\dagger$) operators satisfying the canonical commutator relations (CCR)

$$[a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0 \tag{1}$$

and a vacuum state $|0\rangle$ defined by $a|0\rangle = 0$. The number states are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle, \qquad n = 0, 1, 2, \ldots \tag{2}$$

with ladder relations

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{3}$$

We will use the product-rule commutator identity for operators

$$[A, BC] = [A, B]C + B[A, C] \tag{4}$$

3 Baker-Campbell-Hausdorff (BCH) Factorization

If two operators A and B commute with $[A, B]$ (equivalently: $[A, [A, B]] = [B, [A, B]] = 0$), then

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{5}$$

a special case of the Baker–Campbell–Hausdorff theorem. (For more details on BCH and its proof see the corresponding primer and [4, 5])

4 Hadamard Lemma

The Hadamard lemma in adjoint form (see also BCH primer and [4, 5]) is

$$e^{G} B e^{-G} \;=\; e^{\mathrm{ad}_G} B \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\, \mathrm{ad}_G^{n}(B), \qquad \mathrm{ad}_G(B) \equiv [G, B] \tag{6}$$

where

$$\begin{aligned} \mathrm{ad}_G^0(B) &= B \\ \mathrm{ad}_G^1(B) &= [G, B] \\ \mathrm{ad}_G^2(B) &= [G, [G, B]] \\ &\;\vdots \end{aligned} \tag{7}$$

are nested commutators.

5 Coherent States From The Displacement Generator

The displacement operator is defined as [3]:

$$D(\alpha) = \exp\big(\alpha a^{\dagger} - \alpha^{*} a\big), \qquad \alpha \in \mathbb{C} \tag{8}$$

and the coherent state is the displaced vacuum

$$|\alpha\rangle = D(\alpha)\,|0\rangle \tag{9}$$

Let $A = \alpha a^{\dagger}$ and $B = -\alpha^{*} a$. Using

$$[A, B] = [\alpha a^{\dagger}, -\alpha^{*} a] = -|\alpha|^2 [a^{\dagger}, a] = |\alpha|^2 \tag{10}$$

which is a c-number and therefore trivially commutes with both A and B. We evaluate the operator exponential with the BCH formula (eqn. 5) and obtain the standard factorization

$$D(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha a^{\dagger}}\, e^{-\alpha^{*} a} \tag{11}$$

For completeness and use in later primers we compute $D^{\dagger} a^{\dagger} D$ from the Hadamard lemma (4). First we identify G as the adjoint of the exponent of the displacement operator D:

$$G = (\alpha\, a^{\dagger} - \alpha^{*}\, a)^{\dagger} = \alpha^{*}\, a - \alpha\, a^{\dagger} \tag{12}$$

and evaluating the nested commutators we obtain

$$\begin{aligned} \mathrm{ad}_G^0(a^{\dagger}) &= a^{\dagger} \\ \mathrm{ad}_G^1(a^{\dagger}) &= [G, a^{\dagger}] \\ &= [\alpha^{*}\, a - \alpha\, a^{\dagger}, a^{\dagger}] \\ &= \alpha^{*} \\ \mathrm{ad}_G^2(a^{\dagger}) &= 0 \end{aligned} \tag{13}$$

where the last equation follows uses that any operator commutes with a number. Since only the first two terms remain, we obtain the compact final relationship:

$$D^{\dagger} a^{\dagger} D = a^{\dagger} + \alpha^{*} \tag{14}$$

5.1 Act On The Vacuum And Taylor Expand

Since $a|0\rangle = 0$, every positive power of a annihilates the vacuum, so

$$e^{-\alpha^* a}|0\rangle = \sum_{m=0}^{\infty} \frac{(-\alpha^*)^m}{m!} a^m|0\rangle = |0\rangle \tag{15}$$

Applying (11) then gives

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(a^\dagger)^n|0\rangle \tag{16}$$

Using $(a^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle$ yields the Fock-basis superposition

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle} \tag{17}$$

6 Squeezed Vacuum From The Squeezing Generator

The single-mode squeezing operator is defined by

$$S(\zeta) = \exp\left[\frac{1}{2}\left(\zeta^* a^2 - \zeta (a^\dagger)^2\right)\right], \qquad \zeta \in \mathbb{C} \tag{18}$$

and the squeezed vacuum is

$$|0;\zeta\rangle = S(\zeta)\,|0\rangle, \qquad a|0\rangle = 0 \tag{19}$$

We parameterize ζ in polar form

$$\zeta = re^{i\theta}, \qquad r \geq 0,\ \theta \in \mathbb{R} \tag{20}$$

6.1 Adjoint-Exponential (Hadamard Lemma) And Closed Algebra

Let

$$G \equiv \frac{1}{2}\left(\zeta^* a^2 - \zeta (a^\dagger)^2\right), \qquad S(\zeta) = e^G \tag{21}$$

Using the Hadamard lemma (section 4) we compute the action of $S(\zeta)$ on the ladder operators. From the CCR (eq. 1) we find

$$[G, a] = \zeta\, a^\dagger, \qquad [G, a^\dagger] = \zeta^* a \tag{22}$$

Repeated commutators alternate between a and $a^\dagger$:

$$\mathrm{ad}_G^{2n}(a) = |\zeta|^{2n} a, \qquad \mathrm{ad}_G^{2n+1}(a) = |\zeta|^{2n}\zeta\, a^\dagger \tag{23}$$

with $|\zeta| = r$. Substituting (23) into (6) yields

$$S(\zeta)\,a\,S^\dagger(\zeta) = \sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!}\,a + \sum_{n=0}^{\infty}\frac{r^{2n}}{(2n+1)!}\,\zeta\,a^\dagger$$

$$= a\cosh r + \frac{\zeta}{r}\,a^\dagger\sinh r = a\cosh r + e^{i\theta}a^\dagger\sinh r \tag{24}$$

Define

$$b \equiv S(\zeta)\,a\,S^\dagger(\zeta) = \mu a + \nu a^\dagger, \qquad \mu = \cosh r,\quad \nu = e^{i\theta}\sinh r \tag{25}$$

so that $\nu/\mu = e^{i\theta}\tanh r$. Eqn. 25 is also called *Bogoliubov (linear canonical) transformation* of mode operators.

6.2 Vacuum Condition And Fock-Basis Recursion

Because $|0;\zeta\rangle = S(\zeta)|0\rangle$ and $a|0\rangle = 0$, we have

$$b\,|0;\zeta\rangle = S(\zeta)\,a\,S^\dagger(\zeta)\,S(\zeta)|0\rangle = S(\zeta)\,a\,|0\rangle = 0 \tag{26}$$

Equivalently,

$$(\mu a + \nu a^\dagger)\,|0;\zeta\rangle = 0 \tag{27}$$

Expand the state in the number basis,

$$|0;\zeta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{28}$$

Using $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ in (27), then collecting coefficients of $|n\rangle$, gives the recursion

$$\mu\,c_{n+1}\sqrt{n+1} + \nu\,c_{n-1}\sqrt{n} = 0 \qquad (n \geq 0) \tag{29}$$

with $c_{-1} \equiv 0$. Setting $n = 0$ yields $c_1 = 0$, and (29) then implies $c_{2n+1} = 0$ for all n: only even photon numbers appear.

For even coefficients, set $n = 2k+1$ in (29) to obtain

$$c_{2k+2} = -\frac{\nu}{\mu}\sqrt{\frac{2k+1}{2k+2}}\,c_{2k} \tag{30}$$

Iterating (30) gives

$$c_{2n} = c_0\left(-\frac{\nu}{\mu}\right)^n\frac{\sqrt{(2n)!}}{2^n\,n!} = c_0\big(-e^{i\theta}\tanh r\big)^n\frac{\sqrt{(2n)!}}{2^n\,n!} \tag{31}$$

Normalization fixes c_0. Using the identity

$$\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\,x^n = \frac{1}{\sqrt{1-x}}, \qquad |x| < 1 \tag{32}$$

with $x = \tanh^2 r$, one finds $\sum_{n\geq 0}|c_{2n}|^2 = 1$ implies $|c_0|^2 = 1/\cosh r$. Choosing $c_0 > 0$, the squeezed-vacuum superposition is

$$\boxed{|0;\zeta\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n\,n!}\big(-e^{i\theta}\tanh r\big)^n\,|2n\rangle} \tag{33}$$

7 Energy And Truncation Considerations

The Fock-basis expansions derived above are *infinite* superpositions. Physically, the relevant resource is the field energy, which for a single mode is proportional to the photon number,

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) = \hbar\omega \left(n + \frac{1}{2} \right) \tag{34}$$

so that the mean energy is set by the mean photon number,

$$\langle H \rangle = \hbar\omega \left(\langle n \rangle + \frac{1}{2} \right) \tag{35}$$

An infinite superposition in the Fock basis does *not* automatically imply infinite energy: coherent states and squeezed vacuum states have exponentially suppressed tails in n and remain normalizable with finite $\langle n \rangle$ (e.g. $\langle n \rangle = |\alpha|^2$ for $|\alpha\rangle$ and $\langle n \rangle = \sinh^2 r$ for $|0; \zeta\rangle$). However, any *exact* realization of these ideal states would require perfect unitary control and, in principle, access to arbitrarily high photon-number components, even if their probabilities are small. In practice, source imperfections, losses, detector dynamic range, and circuit nonlinearities impose an effective photon-number cutoff [4].

This motivates the common modeling step of *Hilbert-space truncation*: one chooses a cut-off N_{cut} and replaces the ideal state by its projection onto the finite subspace $\text{span}\{|0\rangle, \ldots, |N_{\text{cut}}\rangle\}$,

$$|\psi\rangle \;\mapsto\; |\psi\rangle_{\text{cut}} = \frac{P_{N_{\text{cut}}}|\psi\rangle}{\sqrt{\langle\psi|P_{N_{\text{cut}}}|\psi\rangle}}, \qquad P_{N_{\text{cut}}} = \sum_{n=0}^{N_{\text{cut}}} |n\rangle\langle n| \tag{36}$$

The truncation error is conveniently quantified by the *tail probability*

$$\epsilon_{\text{tail}}(N_{\text{cut}}) = 1 - \langle\psi|P_{N_{\text{cut}}}|\psi\rangle = \sum_{n > N_{\text{cut}}} |\langle n|\psi\rangle|^2 \tag{37}$$

which directly measures how much probability is discarded. A small ϵ_{tail} is necessary for accurate numerical simulation, but it is also a physical design metric: optical components and detectors must operate reliably over the photon-number range that carries non-negligible weight for the intended α or r.

8 Conclusion and Outlook.

This primer showed how two central classes of single-mode optical states arise directly from their generator forms. For coherent light, the displacement operator can be factorized using a simple BCH identity (made possible because the relevant commutator is a c-number), and acting on the vacuum immediately produces the familiar Poisson-weighted Fock superposition. For squeezed vacuum, the Hadamard lemma shows that conjugation by the quadratic generator closes on the $a, a\dagger$ operator subspace, and the adjoint Taylor series sums to $cosh(r)$ and $sinh(r)$. The resulting linear transformation implies a vacuum condition that yields a simple Fock-basis recursion and the even-photon expansion. Beyond providing explicit expansions useful for calculations and intuition, these derivations highlight a recurring theme in quantum optics [3]: when commutators close on a small operator algebra, exponential generators become tractable and lead to compact, physically transparent results.

References

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