Description:

Density Operators.

$$X = A \otimes B$$

$\{|\alpha_i\rangle;\ i = 0,\dots,2^{m}-1\}$ $\quad\quad$ $\{|\beta_i\rangle;\ i = 0, 2^{n-m}-1\}$

$$\boxed{\begin{array}{l}\text{general state:}\\ |x\rangle = \sum\limits_{i=0;j=0}^{2^m-1,\ 2^{n-m}} x_{ij}\,|\alpha_i\rangle|\beta_j\rangle\end{array}}$$

measure in Subsystem A alone:

$\mathcal{O}^A \otimes \mathbb{1}^B$ $\leftarrow$ projector on total system

probability to measure a outcome in A:

$$p = \langle x|\,\hat{\mathcal{O}}^A \otimes \mathbb{1}^B\,|x\rangle$$

$$= \sum \bar{x}_{ij}\, x_{kl} \langle\alpha_i|\langle\beta_j|\,\hat{P}^A \otimes \mathbb{1}^B\,|\alpha_k\rangle|\beta_l\rangle$$

$\langle\beta_j|\beta_l\rangle = \delta_{jl}$

$$= \sum_{ijk} \bar{x}_{ij}\, x_{kj}\, \langle\alpha_i|\hat{P}^A|\alpha_k\rangle$$

Comment:

Subspace of B no ~~longer~~ longer appears except in the sum.

after some more manipulation: $\sum |\alpha_u\rangle\langle\alpha_u| = \mathbb{1}^A$

$$\langle x|P^A \otimes I|x\rangle = \sum_{ijk} \overline{x_{ij}} x_{kj} \langle\alpha_i|P^A|\alpha_k\rangle$$

$$= \sum_{ik}\sum_{j} \overline{x_{ij}} x_{kj} \langle\alpha_i|P^A \left(\sum_u |\alpha_u\rangle\langle\alpha_u|\right) |\alpha_k\rangle$$

$$= \sum_u \sum_{ik}\sum_j \overline{x_{ij}} x_{kj} \langle\alpha_u|\alpha_k\rangle\langle\alpha_i|P^A|\alpha_u\rangle$$ exchange order.

$$= \sum_u \langle\alpha_u| \left(\sum_{ik}\sum_j \overline{x_{ij}} x_{kj} |\alpha_k\rangle\langle\alpha_i|P^A\right) |\alpha_u\rangle$$

$$= \mathbf{tr}(\rho_x^A P^A),$$

where we define

$$\rho_x^A = \sum_{ik}\sum_j \overline{x_{ij}} x_{kj} |\alpha_k\rangle\langle\alpha_i| \tag{10.2}$$

and call ρ_x^A the *density operator* for $|x\rangle$ on subsystem A. By box 10.3,

$$\rho_x^A = \mathbf{tr}_B(|x\rangle\langle x|). \tag{10.3}$$

"$\sum_j$" trace over all states in subsystem B.

Makes sense?

ρ_x^A : independent of basis

Example: pure state

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

A B

density operator complete space:

$$\rho_{\phi^+} = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right.$$

$$\longrightarrow \frac{1}{2} \begin{array}{c c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

density operator subsystem A:

$$\rho^A_{\phi^+} = tr_B\left(|\phi^+\rangle\langle\phi^+|\right)$$

$$\rho^A_{\phi^+}\Big|_{00} = tr_B\left(\rho_{\phi}^+\right)\Big|_{00}$$

$$= \sum_{\substack{j=0 \\ \uparrow \\ \text{"Bob"}}}^{1} \langle 0|\langle j|\, \phi^+\rangle\langle\phi^+|0\rangle|j\rangle$$

$$= \langle 00|\phi^+\rangle\langle\phi^+|00\rangle + \langle 01|\phi^+\rangle\langle\phi^+|01\rangle$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

$$a_{00} = \sum_{j=0}^{1} \langle 0| \langle j|\, |\psi\rangle\langle\psi|\, |0\rangle |j\rangle = \left(\frac{1}{2}+0\right) = \frac{1}{2},$$

$\langle 0|\,|0\rangle$

$$a_{01} = \sum_{j=0}^{1} \langle 0| \langle j|\, |\psi\rangle\langle\psi|\, |1\rangle |j\rangle = (0+0) = 0,$$

$$a_{10} = \sum_{j=0}^{1} \langle 1| \langle j|\, |\psi\rangle\langle\psi|\, |0\rangle |j\rangle = (0+0) = 0,$$

$$a_{11} = \sum_{j=0}^{1} \langle 1| \langle j|\, |\psi\rangle\langle\psi|\, |1\rangle |j\rangle = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

So

$$\rho_\psi^A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By symmetry, the density operator for Bob's qubit is

$$\rho_\psi^B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

but also no longer unique: loss of information.

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$\Rightarrow \rho_{\psi^+}^A = \rho_\psi^A$$

Caution:

superposition: $=$ $\sum x_i |x_i\rangle$ pure state

mixed state: $=$ $\{\ \text{pure states}\}$

Density operators:

$$|x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

mixed state

$$\rho_{\text{mixed}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} \leftarrow \text{state 1} \\ \leftarrow \text{state 2} \end{matrix}$$

superposition:

$$\varrho_{super} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ different.

what does density operators have to do with entanglement?

~~they~~ contain information on all possible ~~outcomes~~ states including entangled states.

<u>Neumann entropy:</u>

$$S(\varrho) = -\mathrm{tr}(\varrho \log_2 \varrho) = -\sum_i \lambda_i \log_2 \lambda_i$$

general basis eigenbasis

$\varrho_{EPR} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow S(\varrho) = 1$

(n-qubits $S(\varrho)_{max} = n$)

<u>and for pure states:</u>

$$\rho = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$ Only one diagonal element 1

$\Rightarrow S(\varrho) = 0.$

~~In gen.~~

and: entanglement measure $S(\rho)$
does not increase for operator
on single subsystem

$$\mathcal{O}^A \otimes \mathbb{1}^B \quad \text{or} \quad \mathbb{1}^A \otimes \mathcal{O}^B$$

Bipartite systems:

pure state: $X = A \otimes B$

quantum — environment

A, B superpositions, everything can interact with everything

$\Rightarrow$ complicated problem!

A, B can be multi qubit systems themselves.

Huge space:

Simplify by defining equivalence:

L O C C

local operations classical communication

if sequence of unitary operators with

$|\phi\rangle \longrightarrow |\psi\rangle$: equivalent. $\cong$

Example:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \cong |00\rangle$$

measure qubit-1:

$$|\phi^+\rangle \begin{cases} \to |00\rangle \\ \to |11\rangle \xrightarrow{\text{measure X on qubit 2}} \end{cases} = |00\rangle$$

LOCC:

$$|\psi\rangle \xrightarrow{\text{pure}} |\phi\rangle$$

- only transform in subsystem.
- including measurement.
- only classical communication.

→ constraint transformations.

- decrease entanglement possible.

- $S_N(\psi) = S_N(\phi)$ if $|\psi\rangle \underset{LOCC}{\cong} |\phi\rangle$

- "acts on subspace."

<u>other measures:</u>

- Schmid decomposition

- distillable entanglement.

- relative entropy.

Multipartite:

$A \otimes A \otimes B \otimes \cdots$

~~pure states - three qubit system~~

stochastic LOCC = SLOCC

non-zero probability to obtain final state before we had to generate the <u>exact</u> state.

~~3-qubit pure systems~~

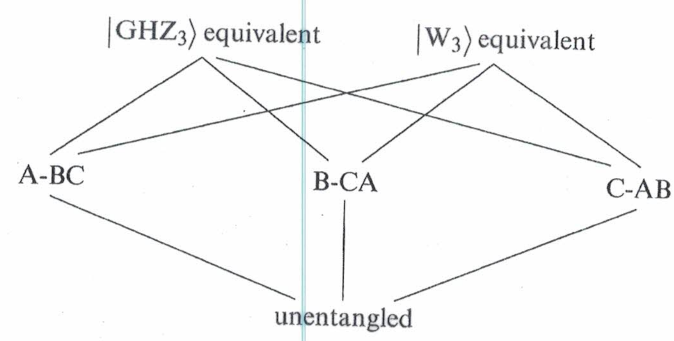
2-qubit system: two classes

- entangled
- unentangled

3-qubit system: six classes.

- unentangled
- A - BC
- B - AC
- C - AB
- SLOCC $|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
- SLOCC $|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

Classification:



Types of entanglement:

Persistency:

$$|\psi\rangle \in V \otimes \cdots \otimes V$$

minimum # of bit to be measured to reach unentangled state.

Maximal connectedness:

if for every 2 qubits we have a sequence of single qubit measurements on the other qubits lead to a maximally ~~[illegible]~~ entangled state for the ~~[illegible]~~ 2 qubits.

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}}\left(|0\cdots0\rangle + |1\cdots1\rangle\right)$$ [two states]

persistency only 1 not very entangled.

$$= \frac{1}{\sqrt{2}}\left(|00\rangle|0\cdots0\rangle + |11\rangle\underbrace{|1\cdots1\rangle}\right)$$

measure ~~[illegible]~~
set to zero

$$= \frac{1}{\sqrt{2}}\underbrace{\left(|00\rangle + |11\rangle\right)}_{\text{maximally entangled}}|0\cdots0\rangle$$

$$|W_n\rangle = \frac{1}{\sqrt{n}}\left(|0\cdots 01\rangle + |0\cdots 010\rangle + \cdots |100\cdots 0\rangle \right)$$

exactly one element $\neq 0$

- not maximally connected
- persistency $= n-1$

$\Rightarrow$ entanglement depends on properties of entanglement we are interested in.

Measurement of density operators

measure observable $\hat{O}$ with k projectors: $\hat{P}_j$

$|x\rangle \in 2^n$ dimensional vectorspace. X

with $\rho = |x\rangle\langle x|$

measurement means projecting:

$$p_j = \langle x|\hat{P}_j|x\rangle$$

$$|x'\rangle \Rightarrow \frac{\hat{P}_j|x\rangle}{|P_j|x\rangle|} = \frac{1}{\sqrt{p_j}}\hat{P}_j|x\rangle$$

Associated density operator for each outcome

$$\rho_x^j = \frac{1}{p_j} \hat{P}_j |x\rangle\langle x| \hat{P}_j^\dagger$$

$$= \frac{1}{p_j} \hat{P}_j \rho_x \hat{P}_j^\dagger$$

$$\Rightarrow \quad \rho_x^{\hat{O}} = \sum_{j=1}^{k} p_j \rho_x^j = \sum_{j=1}^{k} \hat{P}_j \rho_x \hat{P}_j^\dagger = \sum_{j=1}^{k} \hat{P}_j |x\rangle\langle x| \hat{P}_j^\dagger$$

pure states

mixed states = probabilistic mixture over pure states $|\psi_i\rangle$

$$\cancel{\Rightarrow \rho = \sum q_i |\psi_i\rangle\langle\psi_i|}$$

$$\Rightarrow \quad \rho' = \sum q_i \sum \hat{P}_j |\psi_i\rangle\langle\psi_i| \hat{P}_j^\dagger = \sum \hat{P}_j \rho \hat{P}_j^\dagger .$$

$\Rightarrow$ mixtures & pure states are ~~ep~~ described by density matrices.

So far: mainly pure states (chapter 1–9)

subsystems. → density operators.

Decoherence = environment effect (Dynamics)

unitary operators → superoperators.

superoperators create entanglement between

A and B : $A \otimes B$

$$\hat{U}_{AB} |\psi\rangle = \hat{U}_{AB} |\psi\rangle_A \otimes |\psi_B\rangle_B$$

decompose in subsystem B

$$\mathbb{1}_A \otimes \mathbb{1}_B = \mathbb{1}_A \otimes \sum |m\rangle_B {}_B\langle m|$$

$$= \mathbb{1}_A \otimes \sum |m\rangle_B {}_B\langle m| \hat{U}_{AB} | \left(|\psi\rangle_A \otimes |\psi\rangle_B \right)$$

↑

acts on B

$${}_B\langle m|_A \hat{U}_{AB} |\psi_A\rangle \otimes |\psi_B\rangle \in A$$

$$\cancel{|\psi_A\rangle \otimes {}_B\langle m| \hat{U}_{AB} |\psi_B\rangle}$$

$$\Rightarrow \hat{U}_{AB}\left(|\psi\rangle_A \otimes |\psi\rangle_B\right) = \sum_m \hat{A}_m |\psi\rangle_A \otimes |\psi\rangle_B$$

$\Rightarrow$ creates entanglement.

(in contrast to LOCC)

$\hat{U}_{AB}$ = unitary $\Rightarrow$ reversible

$\Rightarrow$ can be back calculated.

$\hat{U}_{AB}$ ≠ not unitary $\Rightarrow$ irreversible.

computational system insufficient.

$\Rightarrow$ measure to "reset" system

or entangle with auxiliary

qubits.

$\Rightarrow$ important class of operators that finds applications in chapter 11:

error correction.